FP3 Complex Numbers

1. June 2010 qu.3

In this question, w denotes the complex number $\cos\frac{2}{5}\pi + \mathrm{i}\sin\frac{2}{5}\pi$.

(i) Express w^2, w^3 and w^* in polar form, with arguments in the interval $0 \le \theta < 2\pi$. [4]

(ii) The points in an Argand diagram which represent the numbers

$$1, \quad 1 + w, \quad 1 + w + w^2, \quad 1 + w + w^2 + w^3, \quad 1 + w + w^2 + w^3 + w^4$$

are denoted by A, B, C, D, E respectively. Sketch the Argand diagram to show these points and join them in the order stated. (Your diagram need not be exactly to scale, but it should show the important features.) [4]

(iii) Write down a polynomial equation of degree 5 which is satisfied by w. [1]

2. June 2010 qu.5

Convergent infinite series C and S are defined by

$$C = 1 + \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \dots,$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \dots.$$

(i) Show that $C + \mathrm{i}S = \dfrac{2}{2 - \mathrm{e}^{\mathrm{i}\theta}}$. [4]

(ii) Hence show that $C = \dfrac{4 - 2\cos\theta}{5 - 4\cos\theta}$, and find a similar expression for S. [4]

3. Jan 2010 qu. 4

(i) Write down, in cartesian form, the roots of the equation $z^4 = 16$. [2]

(ii) Hence solve the equation $w^4 = 16(1 - w)^4$, giving your answers in cartesian form. [5]

4. Jan 2010 qu. 7

(i) Solve the equation $\cos 6\theta = 0$, for $0 < \theta < \pi$. [3]

(ii) By using de Moivre's theorem, show that

$$\cos 6\theta \equiv (2\cos^2\theta - 1)(16\cos^4\theta - 16\cos^2\theta + 1).$$ [5]

(iii) Hence find the exact value of $\cos\left(\frac{1}{12}\pi\right)\cos\left(\frac{5}{12}\pi\right)\cos\left(\frac{7}{12}\pi\right)\cos\left(\frac{11}{12}\pi\right)$,

justifying your answer. [5]

5. June 2009 qu.1

Find the cube roots of $\frac{1}{2}\sqrt{3}+\frac{1}{2}\mathrm{i}$, giving your answers in the form $\cos\theta + \mathrm{i}\sin\theta$, where $0 \le \theta < 2\pi$. [4]

6. June 2009 qu.2

It is given that the set of complex numbers of the form $r\mathrm{e}^{\mathrm{i}\theta}$ for $-\pi < \theta \le \pi$ and $r > 0$, under multiplication, forms a group.

(i) Write down the inverse of $5\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$. [1]

(ii) Prove the closure property for the group. [2]

(iii) Z denotes the element $\mathrm{e}^{\mathrm{i}\gamma}$, where $\frac{1}{2}\pi < \gamma < \pi$. Express Z^2 in the form $\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta < 0$. [2]

7. June 2009 qu.7

(i) Use de Moivre's theorem to prove that $\tan 3\theta \equiv \dfrac{\tan\theta(3-\tan^2\theta)}{1-3\tan^2\theta}$. [4]

(ii) (a) By putting $\theta = \frac{1}{12}\pi$ in the identity in part (i), show that $\tan\frac{1}{12}\pi$ is a solution of the equation $t^3 - 3t^2 - 3t + 1 = 0$. [1]

(b) Hence show that $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$. [4]

(iii) Use the substitution $t = \tan\theta$ to show that $\displaystyle\int_0^{2-\sqrt{3}} \frac{t(3-t^2)}{(1-3t^2)(1+t^2)}\,\mathrm{d}t = a\ln b$,

where a and b are positive constants to be determined. [5]

8. Jan 2009 qu. 2

(i) Express $\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [3]

(ii) Hence find the smallest positive value of n for which $\left(\dfrac{\sqrt{3}+\mathrm{i}}{\sqrt{3}-\mathrm{i}}\right)^n$ is real and positive. [2]

9. Jan 2009 qu. 8

(i) By expressing $\sin\theta$ in terms of $\mathrm{e}^{\mathrm{i}\theta}$ and $\mathrm{e}^{-\mathrm{i}\theta}$, show that

$$\sin^6\theta \equiv -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$ [5]

(ii) Replace θ by $(\frac{1}{2}\pi - \theta)$ in the identity in part (i) to obtain a similar identity for $\cos^6\theta$. [3]

(iii) Hence find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi}(\sin^6\theta - \cos^6\theta)\,\mathrm{d}\theta$. [4]

10. June 2008 qu.4

(i) By expressing $\cos\theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, show that $\cos^5\theta \equiv \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$. [5]

(ii) Hence solve the equation $\cos 5\theta + 5\cos 3\theta + 9\cos\theta = 0$ for $0 \le \theta \le \pi$. [4]

11. June 2008 qu.7

The roots of the equation $z^3 - 1 = 0$ are denoted by 1, ω and ω^2.

(i) Sketch an Argand diagram to show these roots. [1]

(ii) Show that $1 + \omega + \omega^2 = 0$. [2]

(iii) Hence evaluate

(a) $(2 + \omega)(2 + \omega^2)$, [2]

(b) $\dfrac{1}{2+\omega} + \dfrac{1}{2+\omega^2}$. [2]

(iv) Hence find a cubic equation, with integer coefficients,

which has roots 2, $\dfrac{1}{2+\omega}$ and $\dfrac{1}{2+\omega^2}$. [4]

12. Jan 2008 qu. 4

The integrals C and S are defined by

$$C = \int_0^{\frac{1}{2}\pi} e^{2x}\cos 3x\,dx \quad \text{and} \quad S = \int_0^{\frac{1}{2}\pi} e^{2x}\sin 3x\,dx.$$

By considering $C + iS$ as a single integral, show that

$$C = -\frac{1}{13}\left(2 + 3e^{\pi}\right), \quad \text{and obtain a similar expression for } S.$$

(You may assume that the standard result for $\int e^{kx}\,dx$ remains true when k is a complex

constant, so that $\int e^{(a+ib)x}dx = \dfrac{1}{a+ib}e^{(a+ib)x}$ [8]

13. Jan 2008 qu. 7

(i) (a) Verify, without using a calculator, that $\theta = \frac{1}{8}\pi$ is a solution of the equation

$\sin 6\theta = \sin 2\theta$. [1]

(b) By sketching the graphs of $y = \sin 6\theta$ and $y = \sin 2\theta$ for $0 \le \theta \le \frac{1}{2}\pi$ or otherwise,

find the other solution of the equation $\sin 6\theta = \sin 2\theta$ in the interval $0 < \theta < \frac{1}{2}\pi$. [2]

(ii) Use de Moivre's theorem to prove that $\sin 6\theta \equiv \sin 2\theta(16\cos^4\theta - 16\cos^2\theta + 3)$. [5]

(iii) Hence show that one of the solutions obtained in part (i) satisfies $\cos^2\theta = \frac{1}{4}\left(2-\sqrt{2}\right)$, and justify which solution it is. [3]

14. June 2007 qu.1

(i) By writing z in the form $re^{i\theta}$, show that $zz^* = |z|^2$. [1]

(ii) Given that $zz^* = 9$, describe the locus of z [2]

15. June 2007 qu.5

(i) Use de Moivre's theorem to prove that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$. [4]

(ii) Hence find the largest positive root of the equation $64x^6 - 96x^4 + 36x^2 - 3 = 0$,

giving your answer in trigonometrical form. [4]

16. June 2007 qu.7

(i) Show that $(z - e^{i\phi})(z - e^{-i\phi}) \equiv z^2 - (2\cos\phi)z + 1$. [1]

(ii) Write down the seven roots of the equation $z^7 = 1$ in the form $e^{i\theta}$ and show their positions in an Argand diagram. [4]

(iii) Hence express $z^7 - 1$ as the product of one real linear factor and three real quadratic factors. [5]

17. Jan 2007 qu. 3

(i) Solve the equation $z^2 - 6z + 36 = 0$, and give your answers in the form $r(\cos\theta \pm i\sin\theta)$, where $r > 0$ and $0 \le \theta \le \pi$. [4]

(ii) Given that Z is either of the roots found in part (i), deduce the exact value of Z^{-3}. [3]

18. Jan 2007 qu. 8

(i) Use de Moivre's theorem to find an expression for $\tan 4\theta$ in terms of $\tan\theta$. [4]

(ii) Deduce that $\cot 4\theta = \dfrac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$. [1]

(iii) Hence show that one of the roots of the equation $x^2 - 6x + 1 = 0$ is $\cot^2\left(\frac{1}{8}\pi\right)$. [3]

(iv) Hence find the value of $\operatorname{cosec}^2\left(\frac{1}{8}\pi\right) + \operatorname{cosec}^2\left(\frac{3}{8}\pi\right)$, justifying your answer. [5]

19. June 2006 qu.2

(a) Given that $z_1 = 2e^{\frac{1}{6}\pi i}$ and $z_2 = 3e^{\frac{1}{4}\pi i}$, express $z_1 z_2$ and $\frac{z_1}{z_2}$ in the form $re^{i\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [4]

(b) Given that $w = 2(\cos \frac{1}{8}\pi + i \sin \frac{1}{8}\pi)$, express w^{-5} in the form $r(\cos \theta + i \sin \theta)$, where $r > 0$ and $0 \le \theta < 2\pi$. [3]

20. June 2006 qu.7

The series C and S are defined for $0 < \theta < \pi$ by

$$C = 1 + \cos \theta + \cos 2\theta + \cos 3\theta + \cos 4\theta + \cos 5\theta,$$
$$S = \sin \theta + \sin 2\theta + \sin 3\theta + \sin 4\theta + \sin 5\theta.$$

(i) Show that $C + iS = \dfrac{e^{3i\theta} - e^{-3i\theta}}{e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta}} e^{\frac{5}{2}i\theta}$. [4]

(ii) Deduce that $C = \sin 3\theta \cos \frac{5}{2}\theta \operatorname{cosec} \frac{1}{2}\theta$ and write down the corresponding expression for S. [4]

(iii) Hence find the values of θ, in the range $0 < \theta < \pi$, for which $C = S$. [4]

21. Jan 2006 qu. 4

(i) By expressing $\cos \theta$ and $\sin \theta$ in terms of $e^{i\theta}$ and $e^{-i\theta}$, or otherwise, show that

$$\cos^2 \theta \sin^4 \theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$$ [6]

(ii) Hence find the exact value of $\int_0^{\frac{1}{3}\pi} \cos^2 \theta \sin^4 \theta \, d\theta$. [3]

22. Jan 2006 qu. 5

(i) Solve the equation $z^4 = 64(\cos \pi + i \sin \pi)$, giving your answer in polar form. [2]

(ii) By writing your answer to part (i) in the form $x + iy$, find the four linear factors of $z^4 + 64$. [4]

(iii) Hence, or otherwise, express $z^4 + 64$ as the product of two real quadratic factors. [3]